Answer sets in a fuzzy equilibrium logic

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Answer set programming

$$
\begin{array}{lrcl}
r_1: & bad\_weather & \leftarrow & rainy \\
r_2: & bad\_weather & \leftarrow & \sim sunshine \\
r_3: & bbq & \leftarrow & \sim bad\_weather \wedge hungry \\
r_4: & sunshine & \leftarrow & \\
r_5: & hungry & \leftarrow &
\end{array}
$$

Answer set programming

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r_1: & bad\_weather & \leftarrow & rainy \\
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Unique answer set

$$A_1 = \{sunshine, hungry, bbq\}$$

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Fuzzy answer set programming

$$\begin{array}{lrcl}
r_1: & bad_weather & \leftarrow & rainy \\
r_2: & bad_weather & \leftarrow & \sim sunshine \\
r_3: & bbq & \leftarrow & \sim bad_weather \wedge hungry \\
r_4: & sunshine & \leftarrow & 0.8 \\
r_5: & hungry & \leftarrow & 0.7 \\
r_6: & rainy & \leftarrow & 0.1
\end{array}$$

Use degrees of applicability to model continuous phenomena in a logical setting

(no vagueness or uncertainty)

Fuzzy answer set programming

$$
\begin{aligned}
r_1:&\quad bad\_weather &\leftarrow&\quad rainy\\
r_2:&\quad bad\_weather &\leftarrow&\quad \sim sunshine\\
r_3:&\quad bbq &\leftarrow&\quad \sim bad\_weather \wedge hungry\\
r_4:&\quad sunshine &\leftarrow&\quad 0.8\\
r_5:&\quad hungry &\leftarrow&\quad 0.7\\
r_6:&\quad rainy &\leftarrow&\quad 0.1
\end{aligned}
$$

Rule:	accept the head at least to the degree to which the body is true
Logical connectives:	generalize using appropriate $[0,1]^2$-$[0,1]$ mappings

Fuzzy answer set programming

$$
\begin{array}{lrcl}
r_1\colon & bad\_weather & \leftarrow & rainy \\
r_2\colon & bad\_weather & \leftarrow & \sim sunshine \\
r_3\colon & bbq & \leftarrow & \sim bad\_weather \wedge hungry \\
r_4\colon & sunshine & \leftarrow & 0.8 \\
r_5\colon & hungry & \leftarrow & 0.7 \\
r_6\colon & rainy & \leftarrow & 0.1
\end{array}
$$

Unique answer set

$$A_1 = \{sunshine^{0.7}, hungry^{0.7}, bbq^{0.7}, bad\_weather^{0.2}, rainy^{0.1}\}$$

Fuzzy answer set programming

a∧b = min(a,b)

$$
\begin{aligned}
r_1:&\quad bad\_weather &\leftarrow&\quad rainy\\
r_2:&\quad bad\_weather &\leftarrow&\quad (1-0.8)\\
r_3:&\quad bbq &\leftarrow&\quad (1-0.2) \wedge hungry\\
r_4:&\quad sunshine &\leftarrow&\quad 0.8\\
r_5:&\quad hungry &\leftarrow&\quad 0.7\\
r_6:&\quad rainy &\leftarrow&\quad 0.1
\end{aligned}
$$

Unique answer set

$$A_1 = \{sunshine^{0.7}, hungry^{0.7}, bbq^{0.7}, bad\_weather^{0.2}, rainy^{0.1}\}$$

Fuzzy answer set programming

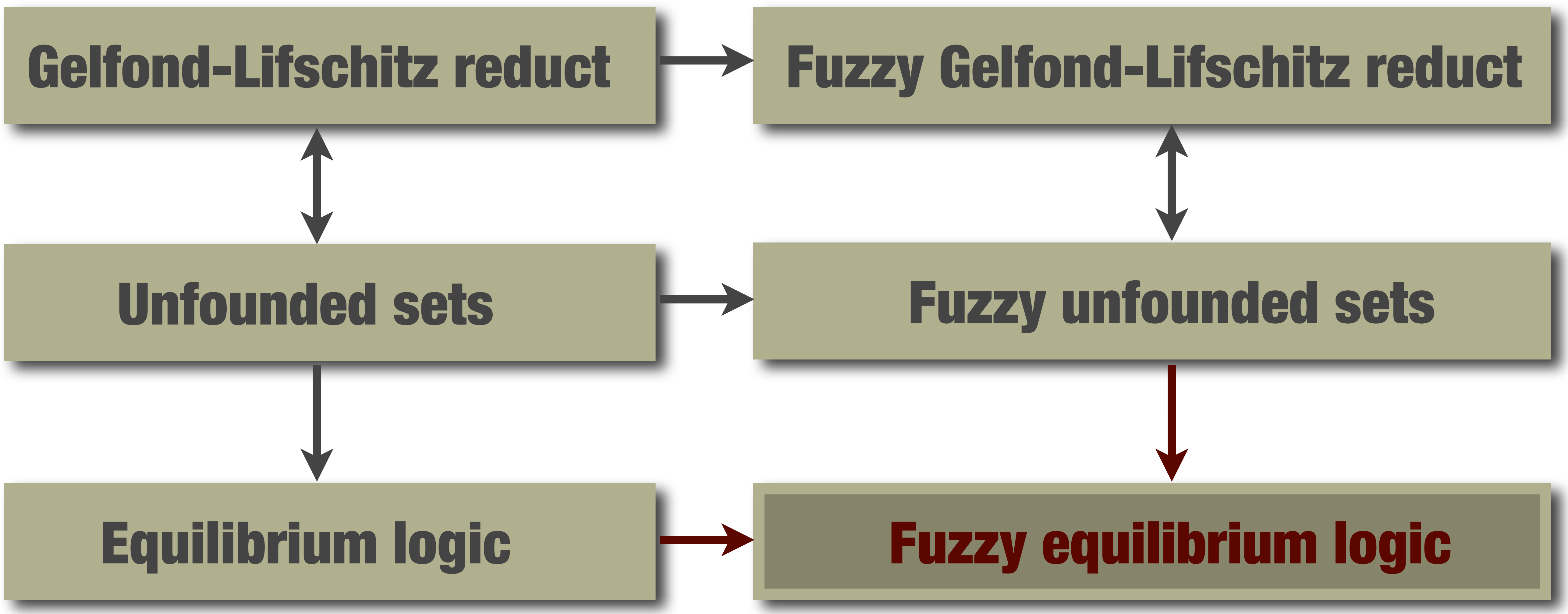
$a \wedge b$ = max(0,a+b-1)

$$
\begin{aligned}
r_1: \quad & bad\_weather & \leftarrow \quad & rainy \\
r_2: \quad & bad\_weather & \leftarrow \quad & (1-0.8) \\
r_3: \quad & bbq & \leftarrow \quad & (1-0.2) \wedge hungry \\
r_4: \quad & sunshine & \leftarrow \quad & 0.8 \\
r_5: \quad & hungry & \leftarrow \quad & 0.7 \\
r_6: \quad & rainy & \leftarrow \quad & 0.1
\end{aligned}
$$

Unique answer set

$$A_1 = \{sunshine^{0.7}, hungry^{0.7}, bbq^{0.5}, bad\_weather^{0.2}, rainy^{0.1}\}$$

Motivation



Many equivalent definitions of answer sets

Equilibrium logic

A valuation V assigns a truth value to atoms in two worlds: **h(ere)** and **t(here)**

$$V(w, a) = \begin{cases} -1 & \text{false} \\ 0 & \text{undecided} \\ 1 & \text{true} \end{cases}$$

The there-world is a refinement of the here-world

$$V(h, a) \neq 0 \Rightarrow V(t, a) = V(h, a)$$

Equilibrium logic

A valuation is extended to arbitrary formulas (h≤h, h≤t, t≤t)

$$\begin{aligned} V(w, \neg\alpha) &= -V(w,\alpha) \\ V(w, \alpha \wedge \beta) &= \min(V(w,\alpha), V(w,\beta)) \\ V(w, \alpha \vee \beta) &= \max(V(w,\alpha), V(w,\beta)) \end{aligned}$$

$$V(w, \alpha \rightarrow \beta) = \begin{cases} 1 & \text{if } \forall w' \geq w\,.\,(V(w',\alpha) = 1) \Rightarrow (V(w',\beta) = 1) \\ -1 & \text{if } V(w,\alpha) = 1 \text{ and } V(w,\beta) = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$V(w, not\,\alpha) = \begin{cases} 1 & \text{if } \forall w' \geq w\,.\,V(w',\alpha) < 1 \\ -1 & \text{if } V(w,\alpha) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Intuition: here-world = what is supported by available rules

there-world = what can be assumed

Equilibrium logic

$$V_h = \{l \in Lit | V(h, l) = 1\} \qquad V_t = \{l \in Lit | V(t, l) = 1\}$$

model

V is a model of a set of formulas, if it makes all formulas true in both worlds.

h-minimal model

V is h-minimal if V_h is minimal over all models V' for which $V'_t = V_t$

equilibrium model

V is an equilibrium model if V is h-minimal and $V_h = V_t$

Fuzzy equilibrium logic

A valuation V assigns a truth value to atoms in two worlds: **h(ere)** and **t(here)**

$$V(w,a) = \begin{cases} [a,a] & \text{true to degree } a \\ [a,b] & \text{undecided (but between } a \text{ and } b\text{)} \end{cases}$$

The there-world is a refinement of the here-world

$$V(t,a) \subseteq V(h,a)$$

Equilibrium logic

A valuation is extended to arbitrary formulas (h≤h, h≤t, t≤t)

$$
\begin{aligned}
V(w, \neg\alpha) &= [1 - V^+(w, \alpha), 1 - V^-(w, \alpha)] \\
V(w, \alpha \otimes \beta) &= [V^-(w, \alpha) \otimes V^-(w, \beta), V^+(w, \alpha) \otimes V^+(w, \beta)] \\
V(w, \alpha \oplus \beta) &= [V^-(w, \alpha) \oplus V^-(w, \beta), V^+(w, \alpha) \oplus V^+(w, \beta)] \\
V(h, \alpha \rightarrow \beta) &= [\min\big(V^-(h, \alpha) \rightarrow V^-(h, \beta), V^-(t, \alpha) \rightarrow V^-(t, \beta)\big), \\
&\quad\; V^-(h, \alpha) \rightarrow V^+(h, \beta)] \\
V(t, \alpha \rightarrow \beta) &= [V^-(t, \alpha) \rightarrow V^-(t, \beta), V^-(t, \alpha) \rightarrow V^+(t, \beta)] \\
V(h, not\, \alpha) &= [1 - V^-(t, \alpha), 1 - V^-(h, \alpha)] \\
V(t, not\, \alpha) &= [1 - V^-(t, \alpha), 1 - V^-(t, \alpha)]
\end{aligned}
$$

Intuition: here-world = what is supported by available rules

there-world = what can be assumed

Equilibrium logic

h-minimal model

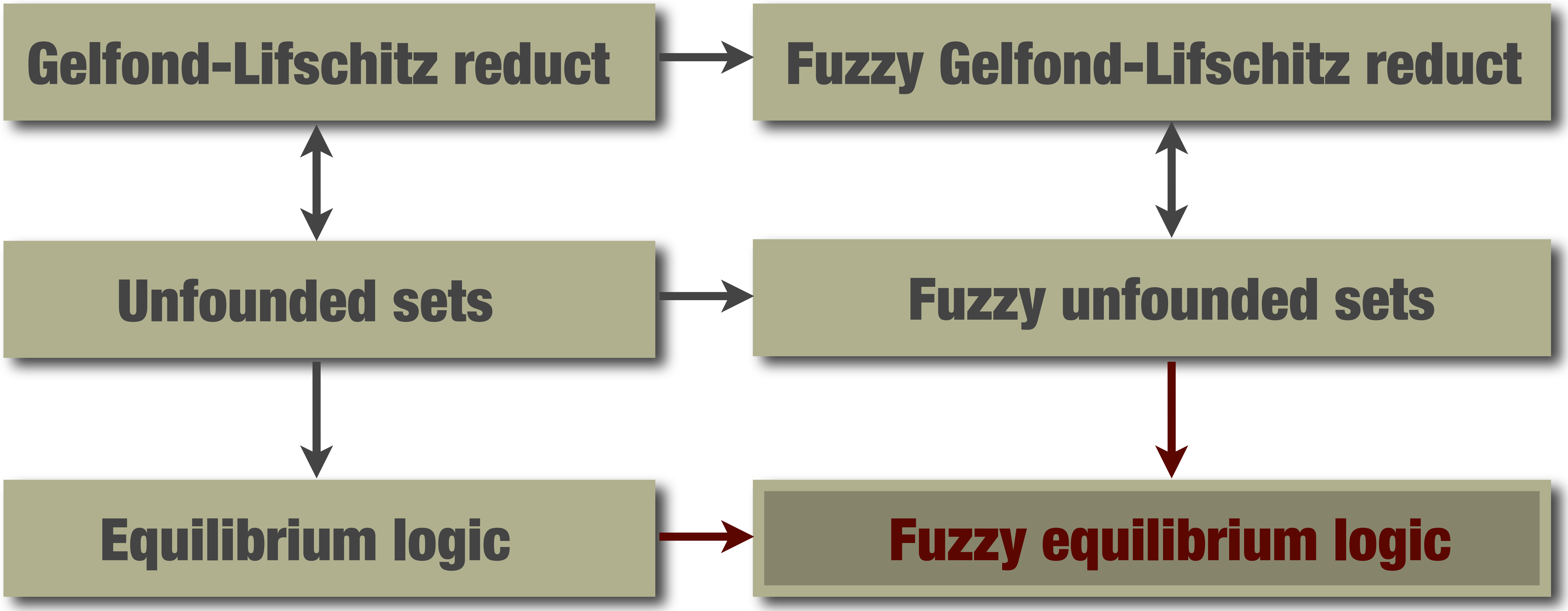
V is h-minimal if V_h is minimal if for all models V' either

1. $V(t,a) \neq V'(t,a)$ *for some* a *in* A; *or*
2. $V'(h,a) \subseteq V(h,a)$ *for all* a *in* A.

equilibrium model

V is an equilibrium model if V is h-minimal
V(h,a)=V(t,a) for all atoms a

Connection with existing approaches



Connection with existing approaches

Classical equilibrium logic

When the syntax is restricted to what can be expressed in classical equilibrium logic, the “fuzzy” equilibrium models coincide with the “classical” equilibrium models when conjunction, disjunction and implication is modeled using the minimum, maximum and Kleene-Dienes implicator

Connection with existing approaches

Gelfond-Lifschitz fuzzy ASP

When the syntax is restricted to what can be expressed in fuzzy ASP definitions based on generalizing the Gelfond-Lifschitz reduct, the equilibrium models coincide with the answer sets

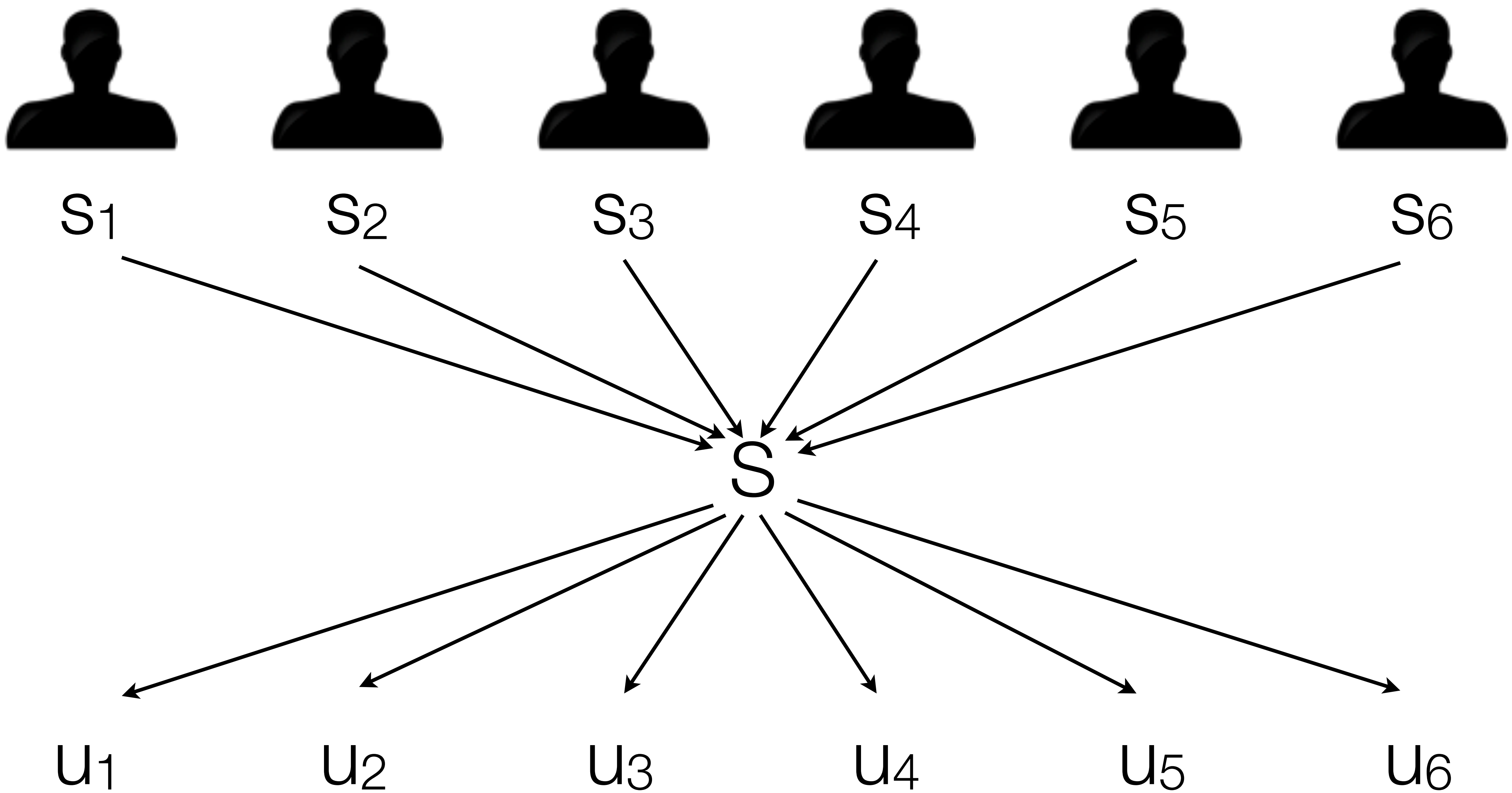
Complexity

Checking whether a given set of formulas Θ in fuzzy equilibrium logic has an equilibrium model is Σ_2^P complete

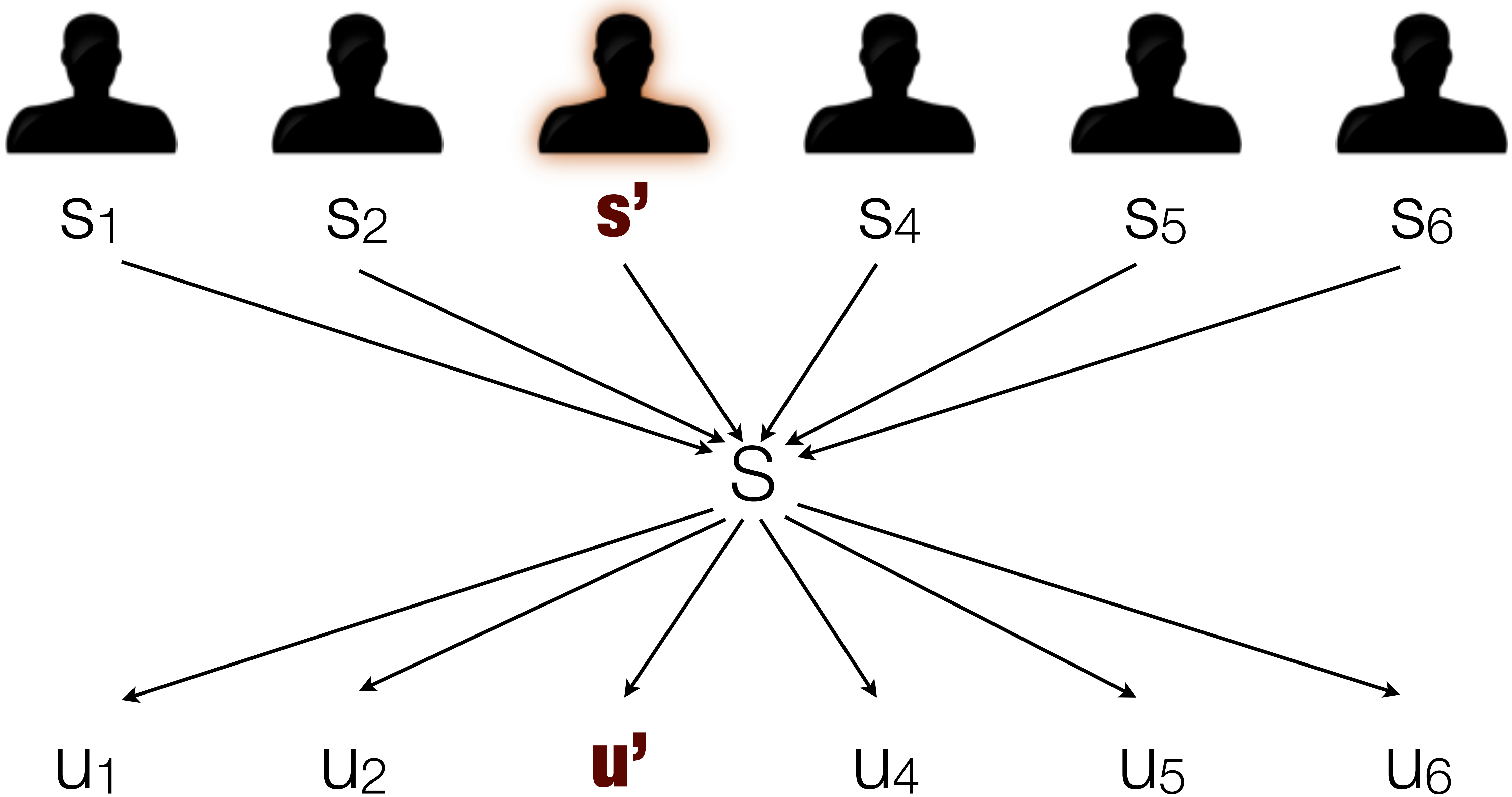
Checking whether the truth value of a given atom is contained in a given interval [a,b] in some equilibrium model of Θ is Σ_2^P complete

Checking whether the truth value of a given atom is contained in a given interval [a,b] in all equilibrium models of Θ is Π_2^P complete

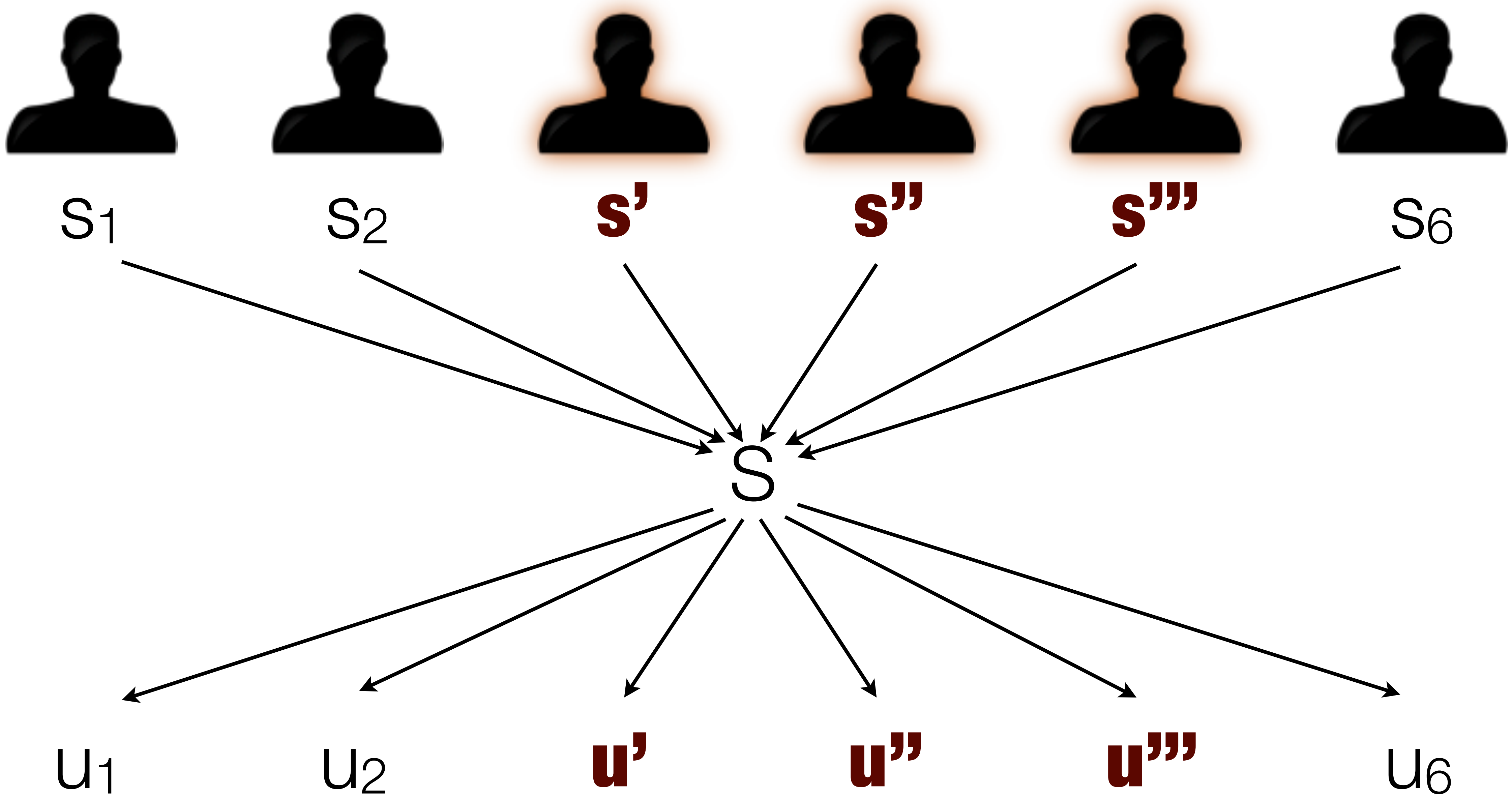
Example: strong pure Nash equilibria



Example: strong pure Nash equilibria



Example: strong pure Nash equilibria



Example: strong pure Nash equilibria

$$a_i \oplus_l \neg a_i$$

- The strategy of player i is represented by a value a_i from [0,1]
- Any strategy may be chosen by any player
- Any valuation in which V(t,ai) = V(h,ai) = [x,x] for some x in [0,1] is an equilibrium model

Guess a strong Nash equilibrium

Example: strong pure Nash equilibria

$$a_i \oplus_l \neg a_i$$
$$c_i^- \oplus_m c_i^+$$
$$d_i^- \oplus_l d_i^+$$

- c_i simulates a "crisp" atom, indicating wether player i is part of the coalition
- For players in the coalition, a new strategy d_i is guessed

Guess a coalition

Example: strong pure Nash equilibria

$$a_i \oplus_l \neg a_i$$
$$c_i^- \oplus_m c_i^+$$
$$d_i^- \oplus_l d_i^+$$
$$e_i^+ \leftarrow_l (a_i \otimes_m c_i^-) \oplus_m (d_i^+ \otimes_m c_i^+)$$
$$e_i^- \leftarrow_l (\neg a_i \otimes_m c_i^-) \oplus_m (d_i^- \otimes_m c_i^+)$$

- $e_i = a_i$ for players outside the coalition
- $e_i = d_i$ for players in the coalition

Guess a coalition

Example: strong pure Nash equilibria

$$
\begin{aligned}
a_i &\oplus_l \neg a_i \\
c_i^- &\oplus_m c_i^+ \\
d_i^- &\oplus_l d_i^+ \\
e_i^+ &\leftarrow_l (a_i \otimes_m c_i^-) \oplus_m (d_i^+ \otimes_m c_i^+) \\
e_i^- &\leftarrow_l (\neg a_i \otimes_m c_i^-) \oplus_m (d_i^- \otimes_m c_i^+) \\
w_i &\leftarrow_l (U_i(e_1^+, ..., e_n^+; e_1^-, ..., e_n^-) \leftarrow_{rs} U_i(a_1, ..., a_n; \neg a_1, ..., \neg a_n))
\end{aligned}
$$

Check whether the new strategies improve the utility of player i

Example: strong pure Nash equilibria

$$
\begin{aligned}
a_i &\oplus_l \neg a_i \\
c_i^- &\oplus_m c_i^+ \\
d_i^- &\oplus_l d_i^+ \\
e_i^+ &\leftarrow_l (a_i \otimes_m c_i^-) \oplus_m (d_i^+ \otimes_m c_i^+) \\
e_i^- &\leftarrow_l (\neg a_i \otimes_m c_i^-) \oplus_m (d_i^- \otimes_m c_i^+) \\
w_i &\leftarrow_l (U_i(e_1^+, ..., e_n^+; e_1^-, ..., e_n^-) \leftarrow_{rs} U_i(a_1, ..., a_n; \neg a_1, ..., \neg a_n)) \\
w &\leftarrow_l ((c_1^+ \otimes_m w_1) \oplus_m ... \oplus_m (c_n^+ \otimes_m w_n)) \oplus_m (c_1^- \otimes_m ... \otimes_m c_n^-)
\end{aligned}
$$

Check whether the new strategies improve the utilities of all players in the coalition

Example: strong pure Nash equilibria

$$
\begin{aligned}
a_i &\oplus_l \neg a_i \\
c_i^- &\oplus_m c_i^+ \\
d_i^- &\oplus_l d_i^+ \\
e_i^+ &\leftarrow_l (a_i \otimes_m c_i^-) \oplus_m (d_i^+ \otimes_m c_i^+) \\
e_i^- &\leftarrow_l (\neg a_i \otimes_m c_i^-) \oplus_m (d_i^- \otimes_m c_i^+) \\
w_i &\leftarrow_l (U_i(e_1^+, ..., e_n^+; e_1^-, ..., e_n^-) \leftarrow_{rs} U_i(a_1, ..., a_n; \neg a_1, ..., \neg a_n)) \\
w &\leftarrow_l ((c_1^+ \otimes_m w_1) \oplus_m ... \oplus_m (c_n^+ \otimes_m w_n)) \oplus_m (c_1^- \otimes_m ... \otimes_m c_n^-) \\
d_i^- &\leftarrow_l w \\
d_i^+ &\leftarrow_l w \\
c_i^- &\leftarrow_l w \\
c_i^+ &\leftarrow_l w \\
w_i &\leftarrow_l w \\
0 &\leftarrow_l \mathit{not}\, w
\end{aligned}
$$

Concluding remarks

- We defined fuzzy equilibrium logic as a generalization of
  - Equilibrium logic
  - Fuzzy answer set programming
- Complexity is the same as for classical equilibrium logic
- Provides a convenient way to encode problems over continuous domains, which are at the second level of the polynomial hierarchy
- Future work: implementation using bi-level mixed integer programming
- Future work: look at theoretical benefits (e.g. checking strong equivalence)